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A property of random context picture grammars

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Abstract

We use random context picture grammers to generate pictures through successive refinement. The productions of such a grammar are context free, but their application is regulated by context randomly distributed in the developing picture. Grammars using this relatively weak context often succeed where context-free grammars fail, e.g., in generating the typical iteration sequence of the Sierpiński carpet. On the other hand, it proved possible to develop iteration theorems for three subclasses of these grammars; finding necessary conditions is problematic in the case of most models of context-free picture grammars with context-sensing ability, since they consider a variable and its context as a connected unit.

We present a property of all picture sets generated with random context picture grammers, and then construct a picture set that does not belong to this class.

Keywords: Formal languages; Picture grammars; Regulated rewriting; Random context grammars; Property

1. Introduction

Syntactic methods of picture generation have become established during the last decade or two. A variety of methods exists and extensive lists of references can be found [7,8].

We introduced random context picture grammars (rcpgs) [5], which are context-free grammars with regulated rewriting, i.e., a production is context-free, but its application may depend on context randomly distributed in the developing picture. The prime

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characteristic of rcpgs is that the location of the context in the developing picture is irrelevant; merely the presence or absence of context matters. Naturally, this imposes limitations on the sets of pictures that can be generated by these grammars. In order to understand these, we have studied three natural subclasses of the class of all rcpgs. For context-free grammars we gave a pumping–shrinking lemma and used it [5] to show that they are strictly weaker than both the random permitting and random forbidding context grammars. For grammars which use permitting context only, we presented a pumping lemma [3] and showed that they are strictly weaker than rcpgs. For grammars which use forbidding context only, we presented a shrinking lemma [2] and showed [4] that they too are strictly weaker than rcpgs.

We now consider the class of all rcpgs. First, we study those which generate only pictures that are composed of squares of equal size and show that these picture sets enjoy a certain commutativity. This enables us to construct a set of pictures that cannot be generated by an rcpg. Then we generalise this last result to the class of all rcpgs. In order to do this, we need a slightly more general notion of commutativity than the one introduced first. Intuitively speaking, two pictures are commutative if one can be transformed into the other by interchanging two of its subsquares, if necessary accompanied by a uniform scaling up of one subsquare and a corresponding scaling down of the other. Our main theorem states that if an rcpg generates an infinite set of pictures, then every picture in the set—apart from a finite number—commutes in the above sense with another picture in the set; moreover, the subsquares involved are no smaller than a predetermined minimum which depends on the grammar and not on the picture.

We formally introduce rcpgs in Section 2. In Section 3, we concentrate on pictures that are composed of squares of equal size and construct a set of pictures that cannot be generated by an rcpg. Finally, in Section 4, we give a characterisation of the class of all rcpgs.

2. Definitions

We generate pictures using productions of the form in Fig. 1, where A is a variable, $m \in \{1,2,3,\dots\}$, $x_{11}, x_{12}, \dots, x_{mm}$ are variables or terminals, and $\mathscr{P}$ and $\mathscr{F}$ are sets of variables. The interpretation is as follows. If a developing picture contains a square labelled A and if all variables of $\mathscr{P}$ and none of $\mathscr{F}$ appear as labels of squares in the picture, then the square labelled A may be divided into equal squares with labels $x_{11}, x_{12}, \dots, x_{mm}$.

In order to simplify the formulation, we denote the square with sides parallel to the axes, lower left-hand vertex at (s,t) and upper right-hand vertex at (u,v) by $((s,t),(u,v))$, using lower case Greek letters for such constructs. Thus (A,α) denotes a square α labelled A. Furthermore, if α is such a square, $\alpha_{11}, \alpha_{12}, \dots, \alpha_{mm}$ denote the equal squares into which α can be divided, with, e.g., α_{11} denoting the left bottom one.

A *random context picture grammar* (rcpg) $G = (V_N, V_T, P, (S, \sigma))$ has a finite alphabet V of *labels*, consisting of disjoint subsets V_N of *variables* and V_T of *terminals*. P is a finite set of *productions* of the form $A \to [x_{11}, x_{12}, \dots, x_{mm}]$ $(\mathscr{P}; \mathscr{F})$,

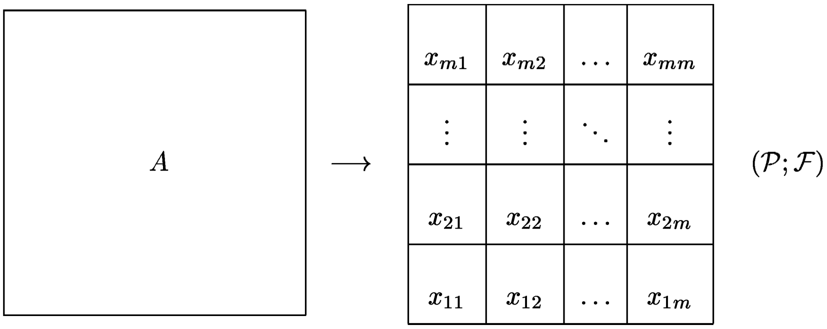

Fig. 1. Production of a random context picture grammar.

where $m \in \{1,2,3,\dots\}$, $A \in V_N$, $x_{11}, x_{12}, \dots, x_{mm} \in V$ and $\mathscr{P}, \mathscr{F} \subseteq V_N$. Finally, there is an *initial labelled square* (S, σ) with $S \in V_N$.

A *pictorial form* is any finite set of nonoverlapping labelled squares in the plane. If Π is a pictorial form, we denote by $l(\Pi)$ the set of labels used in Π. The *size* of a pictorial form Π is the number of squares contained in it, i.e. $|\Pi|$.

For an rcpg G and pictorial forms Π and Γ, we write $\Pi \Rightarrow_G \Gamma$ if there is a production $A \to [x_{11}, x_{12}, \dots, x_{mm}]$ $(\mathscr{P}; \mathscr{F})$ in G, Π contains a labelled square (A, α), $l(\Pi \backslash \{(A,\alpha)\}) \supseteq \mathscr{P}$ and $l(\Pi \backslash \{(A,\alpha)\}) \cap \mathscr{F} = \emptyset$, and $\Gamma = (\Pi \backslash \{(A,\alpha)\}) \cup \{(x_1, \alpha_{11}),$ $(x_2, \alpha_{12}), \dots, (x_{mm}, \alpha_{mm})\}$. As usual, $\Rightarrow_G^*$ denotes the reflexive transitive closure of $\Rightarrow_G$.

A *picture* is a pictorial form Π with $l(\Pi) \subseteq V_T$. The *gallery* $\mathscr{G}(G)$ *generated by a grammar* $G = (V_N, V_T, P, (S, \sigma))$ is the set of pictures Π such that $\{(S, \sigma)\} \Rightarrow_G^* \Pi$.

If every production in a grammar G has $\mathscr{P} = \mathscr{F} = \emptyset$, we call G a *context-free picture grammar* (cfpg); if $\mathscr{F} = \emptyset$ for every production, G is a *random permitting context picture grammar* (rPcpg), and when $\mathscr{P} = \emptyset$, G is a *random forbidding context picture grammar* (rFcpg).

3. The limitation of random context

We now turn to the limitation of rcpgs. First, we study those which generate only pictures that are composed of squares of equal size and show that these picture sets enjoy a certain commutativity. This enables us to construct a picture set that cannot be generated using random context only.

For the sake of simplicity we shall consider only rcpgs of which every production that effects a subdivision produces exactly four subsquares. Also, we let σ be the unit square $((0,0),(1,1))$. The reader will have no trouble to see that the result we are developing here can be formulated for the case of rcpgs with productions that effect other subdivisions.

A picture is called *n-divided*, for $n \geqslant 1$, if it consists of 4^n equal subsquares, each labelled with a terminal. For example, the picture in Fig. 2 is 2-divided. A *level-m subsquare* of an n-divided picture, with $1 \leqslant m \leqslant n$, is a square $((x2^{-m}, y2^{-m}),$

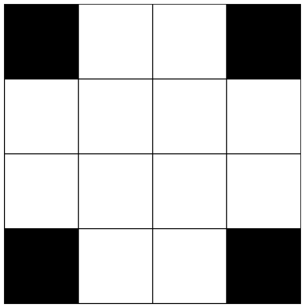

Fig. 2. A 2-divided picture.

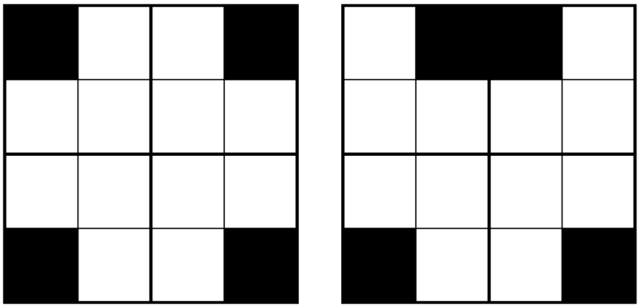

Fig. 3. Two 2-divided pictures commuting at level 1.

$((x+1)2^{-m},(y+1)2^{-m}))$, where x and y are integers and $0 \leqslant x, y < 2^m$. Note that, for $m < n$, a level-m subsquare consists of all 4^{n-m} labelled subsquares contained in it. For example, the upper left-hand quarter of Fig. 2 is a level-1 subsquare of the picture and consists of 4 labelled subsquares.

Two n-divided pictures Φ_1 and Φ_2 are said to *commute at level m* if Φ_1 contains two different level-m subsquares α and β such that Φ_2 can be obtained by simply interchanging the labelling of α and β. This implies that Φ_1 is obtained when the similarly situated subsquares of Φ_2 are interchanged. For example, the two 2-divided pictures in Fig. 3 commute at level 1. A picture Φ_1 is called *self-commutative at level m* if Φ_1 and Φ_1 commute at level m. For example, the 2-divided picture in Fig. 4 is self-commutative at level 1: when the lower left-hand and upper right-hand quarters are interchanged, the picture itself is produced.

Theorem 3.1. *Let $G = (V_{\mathrm{N}}, V_{\mathrm{T}}, P, (S, \sigma))$ be an rcpg that generates an infinite gallery of n-divided pictures. Then there exist an m and a c such that each picture that is c-divided is either self-commutative at level m or commutes with another picture in the gallery at level m.*

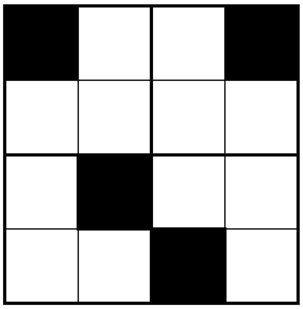

Fig. 4. A 2-divided picture that is self-commutative at level 1.

Proof. As was remarked above, we may assume, without loss of generality, that every production in P that effects a subdivision produces exactly four subsquares. Let $t = |V_N|$ and consider an n-divided picture Φ_1 with $n > t$. In any given derivation of Φ_1, let Π be the first pictorial form that contains a level-t subsquare. Then Π will contain at least $3t + 1$ subsquares, all labelled by variables. This means that there will be a variable, say A, which appears at least twice in Π, say as label of the squares α and β. Then α and β are of equal size, else Π could derive a picture that is not n-divided. So α and β are both level-h subsquares for some $h \leqslant t$. Π then not only derives Φ_1, but also a—not necessarily different—picture Φ_2 with the property that it can be obtained from Φ_1 by simply interchanging the subsquares α and β. Thus we could take $m = t$ and $c = t + 1$. □

Suppose $m \geqslant 1$ is given. Consider the $2m$-divided picture Φ that is constructed as follows. For any level-m subsquare α in Φ, if α is in row i and column j of Φ, then the level-$2m$ subsquare in row i and column j of α is coloured dark. All the other level-$2m$ subsquares are coloured light. Then Φ is not self-commutative at level m. Thus we have.

Theorem 3.2. *There exists a set of pictures, each consisting of the unit square subdivided into equal subsquares and coloured with two colours, that cannot be generated by an rcpg.*

For example, in Fig. 5, $m = 1$. The picture is (2×1)-divided, i.e., 2-divided. We show the level-1 subsquare α in row 1, column 2. The level-2 subsquare in row 1, column 2 of α is coloured dark. All other level-2 subsquares of α are coloured light. The picture cannot be self-commutative at level 1.

Furthermore, in Fig. 6, $m = 2$. The picture is 2×2-divided, i.e., 4-divided. We show the level-2 subsquares α_1 in row 2, column 2, and α_2 in row 3, column 4. The level-4 subsquare in row 2, column 2 of α_1 is coloured dark and all other level-4 subsquares of α_1 are coloured light. Similarly, the level-4 subsquare in row 3, column 4 of α_2 is coloured dark and all other level-4 subsquares of α_2 are coloured light.

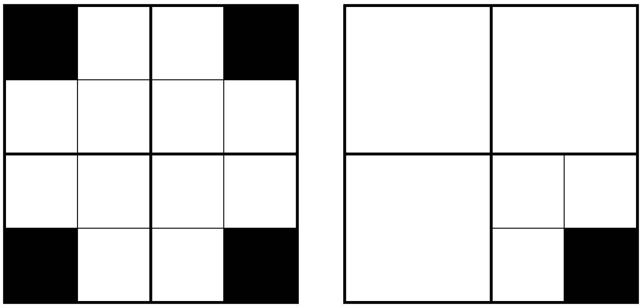

Fig. 5. 2×1-divided.

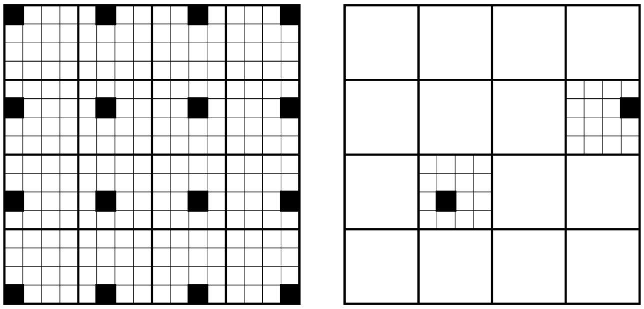

Fig. 6. 2×2-divided.

4. Characteristic of random context galleries

We now generalise Theorem 3.1 to the class of all rcpgs. In order to do this, we need a slightly more general notion of commutativity than the one introduced above. Intuitively speaking, two pictures are commutative if one can be transformed into the other by interchanging two of its subsquares, if necessary accompanied by a uniform scaling up of one subsquare and a corresponding scaling down of the other.

Now let G be an rcpg and suppose $\Phi_1=\{(x_1,\alpha_1),\dots,(x_n,\alpha_n)\}$ is a picture generated by G. Then every α_i is a level-m_i subsquare of $((0,0),(1,1))$ for some $m_i\geqslant 0$. For arbitrary p and q, let α be a level-p subsquare of $((0,0),(1,1))$ such that $\{\alpha_{i_1},\dots,\alpha_{i_s}\}$ represents a tiling of α, and let β be a level-q subsquare of $((0,0),(1,1))$, area-disjoint from α, such that $\{\alpha_{j_1},\dots,\alpha_{j_t}\}$ represents a tiling of β. Let T_1 be the affine mapping that sends α to β, and let T_2 be the inverse mapping. Consider the picture Φ_2 which results from Φ_1 when (x_{i_r},α_{i_r}) is replaced by $(x_{i_r},T_1\alpha_{i_r})$, $r=1,\dots,s$, and (x_{j_r},α_{j_r}) is replaced by $(x_{j_r},T_2\alpha_{j_r})$, $r=1,\dots,t$. Then Φ_1 and Φ_2 are said to be p–q-commutative. For example, the two pictures in Fig. 7 are 1–2-commutative. If it turns out that Φ_1 and Φ_2 are equal, then Φ_1 is called p–q-self-commutative. The left-hand picture in Fig. 7 is 1–2-self-commutative: if we exchange the upper left-hand quarter and the lower right-hand sixteenth, the picture itself results.

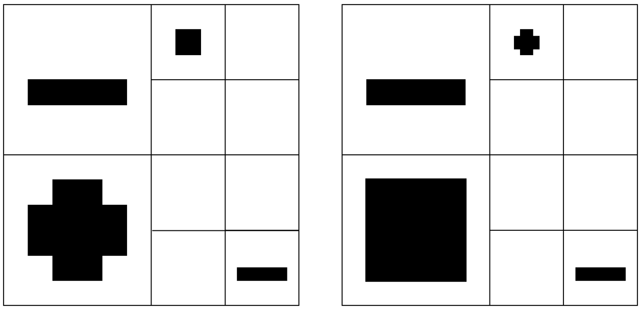

Fig. 7. 1–2-commutative and 1–2-self-commutative pictures.

Theorem 4.1. *Let G be an rcpg that generates an infinite gallery. Then there are numbers c and k such that for every picture Φ_1 with $|\Phi_1| \geqslant c$, there is a picture Φ_2 in the gallery such that Φ_1 and Φ_2 are p–q-commutative for some p and q with $p, q \leqslant k$.*

Proof. Suppose G has s terminals and t variables. Let Φ_1 be any picture in the gallery with $|\Phi_1| > s+t$. Then any derivation of Φ_1 can be written as $\{(S, ((0,0),(1,1)))\} \Rightarrow_G^* \Pi_1 \Rightarrow_G^* \cdots \Rightarrow_G^* \Pi_h \Rightarrow_G^* \Phi_1$, where $|\Pi_i| = 3i+1$ and $|\Phi_1| = 3h+1$. Let j be the smallest index such that $|\Pi_j| > s+t$. Then the smallest subsquare in Π_j is level-j or larger. Now either we can find two subsquares α at level p and β at level q of Π_j which are labelled by the same terminal, or we can find such subsquares labelled by the same variable. In the former case Φ_1 is p–q-self-commutative. In the latter case, it is easy to see that the derivation for Φ_1 can be transformed into a derivation for a picture Φ_2 such that Φ_1 and Φ_2 are p–q-commutative, since the possibility of applying a production is dependent only on the context and not on the position of a variable. The proof is completed by noting that we can choose $c = s+t+1$ and $k = \lceil (s+t)/3 \rceil$. □

5. Future work

There are two obvious directions for future research.

Firstly, the exact relationship between rPcpgs and rFcpgs is not known. As was mentioned in Section 1, it is known that both the rPcpgs and the rFcpgs are strictly more powerful than the cfpgs, and strictly weaker than the rcpgs. Moreover, it is easy to find galleries that can be generated by rFcpgs, but not by any rPcpg [3]. However, it is not known whether there exists a gallery that can be generated by a rPcpg, but not by any rFcpg.

For random context (string) grammars the situation is similar, as can be gleaned from [1,6,10,11].

Secondly, a problem reminiscent of the one solved in this paper is still open for the one-dimensional case. While it is well-known that random context (string) grammars without erasing productions lie strictly between the context-free and context-sensitive grammars [1], no concrete example of a context-sensitive language that cannot be generated by any random context grammar has been found thus far. Initial investigations have indicated that the problem will be more difficult to solve for strings than for pictures [9].

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